Computing the Tutte polynomial of a hyperplane arrangement

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Abstract

We define and study the Tutte polynomial of a hyperplane arrangement. We introduce a method for computing it by solving an enumerative problem in a finite field. For specific arrangements, the computation of Tutte polynomials is then reduced to certain related enumerative questions. As a consequence, we obtain new formulas for the generating functions enumerating alternating trees, labelled trees, semiorders and Dyck paths.

1 Introduction.

Much work has been devoted in recent years to studying hyperplane arrangements and, in particular, their characteristic polynomials. The polynomial $\chi_{\mathcal{A}}(q)$ is a very powerful invariant of the arrangement $\mathcal{A}$; it arises very naturally in many different contexts. Two of the many beautiful results about the characteristic polynomial of an arrangement are the following.

Theorem 1.1 *(Zaslavsky, [45]) Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. The number of regions into which $\mathcal{A}$ dissects $\mathbb{R}^n$ is equal to $(-1)^n\chi_{\mathcal{A}}(-1)$. The number of regions which are relatively bounded is equal to $(-1)^n\chi_{\mathcal{A}}(1)$.*

Theorem 1.2 *(Orlik-Solomon, [24]) Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$, and let $M_{\mathcal{A}} = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}} H$ be its complement. Then the Poincaré polynomial of the cohomology ring of $M_{\mathcal{A}}$ is given by:*

$$\sum_{k\geq 0} \operatorname{rank} H^k(M_{\mathcal{A}}, \mathbb{Z})\, q^k = (-q)^n \chi_{\mathcal{A}}(-1/q).$$

Several authors have worked on computing the characteristic polynomials of specific hyperplane arrangements. This work has led to some very nice enumerative results; see for example [4], [28].

It is somewhat surprising that nothing has been said about the Tutte polynomial of a hyperplane arrangement. Graphs and matroids have a Tutte polynomial associated with them, which generalizes the characteristic polynomial, and

arises very naturally in numerous enumerative problems in both areas. Many interesting invariants of graphs and matroids can be computed immediately from this polynomial.

The present paper, in conjunction with [3], aims to define and investigate the Tutte polynomial of a hyperplane arrangement. This paper is devoted to purely enumerative questions. We are particularly interested in computing the Tutte polynomials of specific arrangements. We address the matroid-theoretic aspects of this investigation in [3].

The paper is organized as follows. In Section 2 we introduce the basic notions of hyperplane arrangements that we will need. In Section 3 we define the Tutte polynomial of a hyperplane arrangement, and we present a finite field method for computing it. This is done in terms of the coboundary polynomial, a simple transformation of the Tutte polynomial. We recover several known results about the characteristic and Tutte polynomials of graphs and representable matroids, and derive other consequences of this method. Finally, in Section 4, we compute the Tutte polynomials of several families of arrangements. In particular, for deformations of the braid arrangement, we relate the computation of Tutte polynomials to the enumeration of classical combinatorial objects. As a consequence, we obtain several purely enumerative results about objects such as labeled trees, Dyck paths, alternating trees and semiorders.

2 Hyperplane arrangements.

In this section we recall some of the basic concepts of hyperplane arrangements. For a more thorough introduction, we refer the reader to [25] or [38].

Given a field $\mathbb{k}$ and a positive integer n, an *affine hyperplane* in $\mathbb{k}^n$ is an $(n-1)$-dimensional affine subspace of $\mathbb{k}^n$. If we fix a system of coordinates $x_1, \ldots, x_n$ on $\mathbb{k}^n$, a hyperplane can be seen as the set of points that satisfy a certain equation $c_1x_1 + \cdots + c_nx_n = c$, where $c_1, \ldots, c_n, c \in \mathbb{k}$ and not all c_i's are equal to 0. A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{k}^n$ is a finite set of affine hyperplanes of $\mathbb{k}^n$. We will refer to hyperplane arrangements simply as *arrangements*. We will assume for simplicity that $\mathbb{k} = \mathbb{R}$ unless explicitly stated, although most of our results extend immediately to any field of characteristic zero.

We will say that an arrangement $\mathcal{A}$ is *central* if the hyperplanes in $\mathcal{A}$ have a non-empty intersection.[1] Similarly, we will say that a subset (or *subarrangement*) $\mathcal{B} \subseteq \mathcal{A}$ of hyperplanes is *central* if the hyperplanes in $\mathcal{B}$ have a non-empty intersection.

The *rank function* $r_{\mathcal{A}}$ is defined for each central subset $\mathcal{B}$ by $r_{\mathcal{A}}(\mathcal{B}) = n - \dim \cap \mathcal{B}$. This function can be extended to a function $r_{\mathcal{A}} : 2^{\mathcal{A}} \to \mathbb{N}$, by defining the rank of a non-central subset $\mathcal{B}$ to be the largest rank of a central subset of $\mathcal{B}$. The *rank* of $\mathcal{A}$ is $r_{\mathcal{A}}(\mathcal{A})$, and it is denoted $r_{\mathcal{A}}$.

Alternatively, if the hyperplane H has defining equation $c_1x_1 + \cdots + c_nx_n = c$, associate its normal vector $v = (c_1, \ldots, c_n)$ to it. Then define $r_{\mathcal{A}}(\{H_1, \ldots, H_k\})$

[1] Sometimes we will call an arrangement *affine* to emphasize that it does not need to be central.

to be the dimension of the span of the corresponding vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$. It is easy to see that these two definitions of the rank function agree. In particular, this means that the resulting function $r_{\mathcal{A}} : 2^{\mathcal{A}} \to \mathbb{N}$ is the rank function of a matroid. We will usually omit the subscripts when the underlying arrangement is clear, and simply write $r(\mathcal{B})$ and r for $r_{\mathcal{A}}(\mathcal{B})$ and $r_{\mathcal{A}}$, respectively.[2]

The rank function gives us natural definitions of the usual concepts of matroid theory, such as independent sets, bases, closed sets, and circuits, in the context of hyperplane arrangements. All of this is done more naturally in the broader context of semimatroids in [3].

To each hyperplane arrangement $\mathcal{A}$ we assign a partially ordered set, called the *intersection poset* of $\mathcal{A}$ and denoted $L_{\mathcal{A}}$. It consists of the non-empty intersections $H_{i_1} \cap \cdots \cap H_{i_k}$, ordered by reverse inclusion. This poset is graded, with rank function $r(H_{i_1} \cap \cdots \cap H_{i_k}) = r_{\mathcal{A}}(\{H_{i_1}, \ldots, H_{i_k}\})$, and a unique minimal element $\hat{0} = \mathbb{R}^n$. We will sometimes call two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ *isomorphic*, and write $\mathcal{A}_1 \cong \mathcal{A}_2$, if $L_{\mathcal{A}_1} \cong L_{\mathcal{A}_2}$.

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{n - r(x)}.$$

where μ denotes the Möbius function [33, Section 3.7] of $L_{\mathcal{A}}$.

Let $\mathcal{A}$ be an arrangement and let H be a hyperplane in $\mathcal{A}$. The arrangement $\mathcal{A} - \{H\}$ (or simply $\mathcal{A} - H$), obtained by removing H from the arrangement, is called the *deletion of H in $\mathcal{A}$*. It is an arrangement in $\mathbb{R}^n$. The arrangement $\mathcal{A}/H = \{H' \cap H \,|\, H' \in \mathcal{A} - H, H' \cap H \neq \emptyset\}$, consisting of the intersections of the other hyperplanes with H, is called the *contraction of H in $\mathcal{A}$*. It is an arrangement in H.

Notice, however, that some technical difficulties can arise. In a hyperplane arrangement $\mathcal{A}$, contracting a hyperplane H may give us repeated hyperplanes H_1 and H_2 in the arrangement $\mathcal{A}/H$. Now say we want to contract H_1 in $\mathcal{A}/H$. In passing to the contraction $(\mathcal{A}/H)/H_1$, the hyperplane H_2 of $\mathcal{A}/H$ becomes the "hyperplane" $H_2 \cap H_1 = H_1$ in the "arrangement" $(\mathcal{A}/H)/H_1$. This is not a hyperplane in H_1, though.

Therefore, the class of hyperplane arrangements, as we defined it, is not closed under deletion and contraction. This is problematic when we want to mirror matroid-theoretic results in this context. There is an artificial solution to this problem: we can consider multisets $\{H_1, \ldots, H_k\}$ of subspaces of vector spaces V, where each H_i has dimension $\dim V - 1$ or $\dim V$. In other words, we allow repeated hyperplanes, and we allow the full space V to be regarded as a "hyperplane", mirroring a loop of a matroid. This class of objects *is* closed under deletion and contraction, but it is somewhat awkward to work with. A better solution is to think of arrangements as members of the class of semimatroids; a class that *is* closed under deletion and contraction, and is more natural matroid-theoretically. We develop this point of view in [3]. However,

[2] There is another natural way to extend $r_{\mathcal{A}}$ to the rank function of a matroid; for more information, see [3].

such issues will be irrelevant in this paper, which focuses on purely enumerative aspects of arrangements.

3 Computing the Tutte polynomial.

In [4], Athanasiadis introduced a powerful method for computing the characteristic polynomial of a subspace arrangement, based on ideas of Crapo and Rota [13]. He reduced the computation of characteristic polynomials to an enumeration problem in a vector space over a finite field. He used this method to compute explicitly the characteristic polynomial of several families of hyperplane arrangements, obtaining very nice enumerative results. As should be expected, this method only works when the equations defining the hyperplanes of the arrangement have integer (or rational) coefficients. Such an arrangement will be called a $\mathbb{Z}-$*arrangement*.

In [29], Reiner asked whether it is possible to use [29, Corollary 3] to compute explicitly the Tutte polynomials of some non-trivial families of representable matroids. Compared to all the work that has been done on computing characteristic polynomials explicitly, virtually nothing is known about computing Tutte polynomials.

In this section, we introduce a new method for computing Tutte polynomials of hyperplane arrangements. Our approach does not use Reiner's result; it is closer to Athanasiadis's method. In fact, Athanasiadis's result [4, Theorem 2.2] can be obtained as a special case of the main result of this section, Theorem 3.3, by setting $t = 0$.

After proving Theorem 3.3, we will present some of its consequences. We will then use it in Section 4 to compute explicitly the Tutte polynomials of several families of arrangements.

3.1 The Tutte and coboundary polynomials.

Definition 3.1 *The* Tutte polynomial *of a hyperplane arrangement* $\mathcal{A}$ *is*

$$T_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (q-1)^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|-r(\mathcal{B})}, \tag{3.1}$$

where the sum is over all central subsets $\mathcal{B} \subseteq \mathcal{A}$.

It will be useful for us to consider a simple transformation of the Tutte polynomial, first considered by Crapo [12] in the context of matroids.

Definition 3.2 *The* coboundary polynomial $\overline{\chi}_{\mathcal{A}}(q,t)$ *of an arrangement* $\mathcal{A}$ *is*

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} q^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|}. \tag{3.2}$$

It is easy to check that

$$\overline{\chi}_{\mathcal{A}}(q,t) = (t-1)^r \, T_{\mathcal{A}}\left(\frac{q+t-1}{t-1}, t\right)$$

and

$$T_{\mathcal{A}}(x,y) = \frac{1}{(y-1)^r}\, \overline{\chi}_{\mathcal{A}}\left((x-1)(y-1), y\right).$$

Therefore, computing the coboundary polynomial of an arrangement is essentially equivalent to computing its Tutte polynomial. Our results can be presented more elegantly in terms of the coboundary polynomial.

3.2 The finite field method.

Let $\mathcal{A}$ be a $\mathbb{Z}-$arrangement in $\mathbb{R}^n$, and let q be a prime power. The arrangement $\mathcal{A}$ induces an arrangement $\mathcal{A}_q$ in the vector space $\mathbb{F}_q^n$. If we consider the equations defining the hyperplanes of $\mathcal{A}$, and regard them as equations over $\mathbb{F}_q$, they define the hyperplanes of $\mathcal{A}_q$.

Say that $\mathcal{A}$ *reduces correctly* over $\mathbb{F}_q$ if the arrangements $\mathcal{A}$ and $\mathcal{A}_q$ are isomorphic. This does not always happen; sometimes the hyperplanes of $\mathcal{A}$ do not even become hyperplanes in $\mathcal{A}_q$. For example, the hyperplane $2x+2y=1$ in $\mathbb{R}^2$ becomes the empty "hyperplane" $0=1$ in $\mathbb{F}_2^2$. Sometimes independence is not preserved. For example, the independent hyperplanes $2x+y=0$ and $y=0$ in $\mathbb{R}^2$ become the same hyperplane in $\mathbb{F}_2^2$.

However, if q is a power of a large enough prime, $\mathcal{A}$ will reduce correctly over $\mathbb{F}_q$. To have $\mathcal{A} \cong \mathcal{A}_q$, we need central and independent subarrangements to be preserved. Cramer's rule lets us rephrase these conditions, in terms of certain determinants (formed by the coefficients of the hyperplanes in $\mathcal{A}$) being zero or non-zero. If we let q be a power of a prime which is larger than all these determinants, we will guarantee that $\mathcal{A}$ reduces correctly over $\mathbb{F}_q$.

Theorem 3.3 *Let $\mathcal{A}$ be a $\mathbb{Z}-$arrangement in $\mathbb{R}^n$. Let q be a power of a large enough prime, and let $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then*

$$q^{n-r}\,\overline{\chi}_{\mathcal{A}}(q,t) \;=\; \sum_{p \,\in\, \mathbb{F}_q^n} t^{h(p)} \tag{3.3}$$

where $h(p)$ denotes the number of hyperplanes of $\mathcal{A}_q$ that p lies on.

Proof. Let q be a power of a large enough prime, so that $\mathcal{A}$ reduces correctly over $\mathbb{F}_q$. For each $\mathcal{B} \subseteq \mathcal{A}$, let $\mathcal{B}_q$ be the subarrangement of $\mathcal{A}_q$ induced by it. For each $p \in \mathbb{F}_q^n$, let $H(p)$ be the set of hyperplanes of $\mathcal{A}_q$ that p lies on. From (3.2) we have

$$\begin{aligned}
q^{n-r}\,\overline{\chi}_{\mathcal{A}}(q,t) &= \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \text{central}}} q^{n-r(\mathcal{B})}(t-1)^{|\mathcal{B}|} \\
&= \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \text{central}}} q^{\dim\cap\mathcal{B}}(t-1)^{|\mathcal{B}|} \\
&= \sum_{\substack{\mathcal{B}_q\subseteq\mathcal{A}_q\\ \text{central}}} |\cap\mathcal{B}_q|\,(t-1)^{|\mathcal{B}_q|} \\
&= \sum_{\substack{\mathcal{B}_q\subseteq\mathcal{A}_q\\ \text{central}}} \sum_{p\,\in\,\cap\mathcal{B}_q} (t-1)^{|\mathcal{B}_q|} \\
&= \sum_{p\,\in\,\mathbb{F}_q^n} \sum_{\mathcal{B}_q\subseteq H(p)} (t-1)^{|\mathcal{B}_q|} \\
&= \sum_{p\,\in\,\mathbb{F}_q^n} (1+(t-1))^{h(p)},
\end{aligned}$$

as desired. □

In principle, Theorem 3.3 only computes $\overline{\chi}_{\mathcal{A}}(q,t)$ when q is a power of a large enough prime. In practice, however, when we compute the right-hand side of (3.3) for large prime powers q, we will get a polynomial function in q and t. Since the left-hand side is also a polynomial, these two polynomials must be equal.

Theorem 3.3 reduces the computation of coboundary polynomials (and hence Tutte polynomials) to enumerating points in the finite vector space $\mathbb{F}_q^n$, according to a certain statistic. This method can be extremely useful when the hyperplanes of the arrangement are defined by simple equations. We will illustrate this in section 4.

We remark that Theorem 3.3 was also obtained independently by Welsh and Whittle [43].

3.3 Special cases and applications.

Now we present some known facts and some new results which follow from the finite field method. We start with two classical theorems which are special cases of Theorem 3.3.

3.3.1 Colorings of graphs

A graph G has a matroid M_G associated to it, called the *cycle matroid* of G. Its Tutte polynomial is equal to the (graph-theoretic) Tutte polynomial of G.

From the point of view of arrangements, the construction is the following. Given a graph G on $[n]$, we associate to it an arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$. It consists of the hyperplanes $x_i = x_j$, for all $1 \leq i < j \leq n$ such that ij is an

edge in the graph G. Then we have that $T_G(q,t) = T_{\mathcal{A}_G}(q,t)$. We can define the coboundary polynomial for a graph like we did for arrangements, and then $\overline{\chi}_G(q,t) = \overline{\chi}_{\mathcal{A}_G}(q,t)$ also.

We shall now interpret Theorem 3.3 in this framework. It is easy to see that the rank of G is equal to $n-c$, where c is the number of connected components of G. Therefore the left-hand side of (3.3) is $q^c\,\overline{\chi}_G(q,t)$ in this case.

To interpret the right-hand side, notice that each point $p \in \mathbb{F}_q^n$ corresponds to a q-coloring of the vertices of G. The point $p = (p_1, \ldots, p_n)$ will correspond to the coloring κ_p of G which assigns color p_i to vertex i. A hyperplane $x_i = x_j$ contains p when $p_i = p_j$. This happens precisely when edge ij is *monochromatic* in κ_p; that is, when its two ends have the same color. Therefore, applying Theorem 3.3 to the arrangement $\mathcal{A}_G$, we recover the following known result:

Theorem 3.4 *([11, Proposition 6.3.26]) Let G be a graph with c connected components. Then*

$$q^c\,\overline{\chi}_G(q,t) = \sum_{\substack{q\text{-colorings}\\ \kappa \text{ of } G}} t^{\mathrm{mono}(\kappa)},$$

where $\mathrm{mono}(\kappa)$ *is the number of monochromatic edges in κ.*

3.3.2 Linear codes

Given positive integers $n \geq r$, an $[n,r]$ *linear code* C over $\mathbb{F}_q$ is an r-dimensional subspace of $\mathbb{F}_q^n$. A *generator matrix* for C is an $r \times n$ matrix U over $\mathbb{F}_q$, the rows of which form a basis for C. It is not difficult to see that the isomorphism class of the matroid on the columns of U depends only on C. We shall denote the corresponding matroid M_C.

The elements of C are called *codewords*. The *weight* $w(v)$ of a codeword is the cardinality of its support; that is, the number of non-zero coordinates of v. The *codeweight polynomial* of C is

$$A(C,q,t) = \sum_{v \in C} t^{w(v)}. \tag{3.4}$$

The translation of Theorem 3.3 to this setting is the following.

Theorem 3.5 *(Greene, [15]) For any linear code C over $\mathbb{F}_q$,*

$$A(C,q,t) = t^n\,\overline{\chi}_{M_C}\left(q, \frac{1}{t}\right).$$

Proof. Let $\mathcal{A}_C$ be the central arrangement corresponding to the columns of U. (We can call it $\mathcal{A}_C$ because, as stated above, its isomorphism class depends only on C.) This is a rank r arrangement in $\mathbb{F}_q^r$ such that $\overline{\chi}_{M_C}(q, \frac{1}{t}) = \overline{\chi}_{\mathcal{A}_C}(q, \frac{1}{t})$. Comparing (3.4) with Theorem 3.3, it remains to prove that

$$\sum_{v \in C} t^{w(v)} = \sum_{p \in \mathbb{F}_q^r} t^{n-h(p)}.$$

To do this, consider the bijection $\phi : \mathbb{F}_q^r \to C$ determined by right multiplication by U. If $u_1, \ldots, u_r$ are the row vectors of U, then ϕ sends $p = (p_1, \ldots, p_r) \in \mathbb{F}_q^r$ to the codeword $v_p = p_1 u_1 + \cdots + p_r u_r \in C$. For $1 \leq i \leq n$, p lies on the hyperplane determined by the $i - th$ column of U if and only if the i-th coordinate of v_p is equal to zero. Therefore $h(p) = n - w(v_p)$. This completes the proof. $\square$

3.3.3 Deletion-contraction

The point of view of Theorem 3.3 can be used to give a nice enumerative proof of the deletion-contraction formula for the Tutte polynomial of an arrangement. Once again, this formula is better understood in the context of semimatroids, as shown in [3]. For the moment, leaving matroid-theoretical issues aside, we only wish to present a special case of it as a nice application.

Proposition 3.6 *Let $\mathcal{A}$ be a hyperplane arrangement, and let H be a hyperplane in $\mathcal{A}$ such that $r_{\mathcal{A}}(\mathcal{A}-H) = r_{\mathcal{A}}$. Then $T_{\mathcal{A}}(q,t) = T_{\mathcal{A}-H}(q,t) + T_{\mathcal{A}/H}(q,t)$.*

Proof. Because there will be several arrangements involved, let $h(\mathcal{B}, p)$ denote the number of hyperplanes in $\mathcal{B}_q$ that p lies on. Then

$$\begin{aligned}
q^{n-r}\,\overline{\chi}_{\mathcal{A}}(q,t) &= \sum_{p \in \mathbb{F}_q^n} t^{h(\mathcal{A},\,p)} \\
&= \sum_{p \in \mathbb{F}_q^n - H} t^{h(\mathcal{A},\,p)} + \sum_{p \in H} t^{h(\mathcal{A},\,p)} \\
&= \sum_{p \in \mathbb{F}_q^n - H} t^{h(\mathcal{A}-H,\,p)} + \sum_{p \in H} t^{h(\mathcal{A}-H,\,p)+1} \\
&= \sum_{p \in \mathbb{F}_q^n} t^{h(\mathcal{A}-H,\,p)} + (t-1) \sum_{p \in H} t^{h(\mathcal{A}-H,\,p)} \\
&= q^{n-r}\,\overline{\chi}_{\mathcal{A}-H}(q,t) + (t-1) q^{(n-1)-(r-1)}\,\overline{\chi}_{\mathcal{A}/H}(q,t).
\end{aligned}$$

We conclude that $\overline{\chi}_{\mathcal{A}}(q,t) = \overline{\chi}_{\mathcal{A}-H}(q,t) + (t-1)\,\overline{\chi}_{\mathcal{A}/H}(q,t)$, which is equivalent to the deletion-contraction formula for Tutte polynomials. $\square$

3.3.4 A probabilistic interpretation

Theorem 3.7 *Let $\mathcal{A}$ be an arrangement and let $0 \leq t \leq 1$ be a real number. Let $\mathcal{B}$ be a random subarrangement of $\mathcal{A}$, obtained by independently removing each hyperplane from $\mathcal{A}$ with probability t. Then the expected characteristic polynomial $\chi_{\mathcal{B}}(q)$ of $\mathcal{B}$ is $q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t)$.*

Proof. We have

$$\begin{aligned} E[\chi_{\mathcal{B}}(q)] &= \sum_{\mathcal{C}\subseteq\mathcal{A}} P[\mathcal{B}=\mathcal{C}]\,\chi_{\mathcal{C}}(q) \\ &= \sum_{\mathcal{C}\subseteq\mathcal{A}} P[\mathcal{B}=\mathcal{C}]\,|\,\mathbb{F}_q^n - \cup\mathcal{C}_q| \\ &= \sum_{p\,\in\,\mathbb{F}_q^n} \sum_{\substack{\mathcal{C}\subseteq\mathcal{A}\\ p\notin\cup\mathcal{C}_q}} P[\mathcal{B}=\mathcal{C}], \end{aligned}$$

where in the second step we have used Athanasiadis's result [4]; that is, the case $t=0$ of Theorem 3.3.

Recall that $H(p)$ denotes the set of hyperplanes in $\mathcal{A}_q$ containing p. Then

$$\begin{aligned} E[\chi_{\mathcal{B}}(q)] &= \sum_{p\,\in\,\mathbb{F}_q^n} P[\mathcal{B}_q\cap H(p)=\emptyset] \\ &= \sum_{p\,\in\,\mathbb{F}_q^n} t^{h(p)}, \end{aligned}$$

which is precisely what we wanted to show. □

3.3.5 A Möbius formula

Theorem 3.8 *For an arrangement $\mathcal{A}$ and an affine subspace x in the intersection poset $L_{\mathcal{A}}$, let $h(x)$ be the number of hyperplanes of $\mathcal{A}$ containing x. Then*

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{x\leq y \text{ in } L_{\mathcal{A}}} \mu(x,y)\, q^{r-r(y)} t^{h(x)}.$$

Proof. Consider the arrangement $\mathcal{A}$ restricted to $\mathbb{F}_q^n$, where q is a power of a large enough prime, so that $\mathcal{A}$ reduces correctly over $\mathbb{F}_q$. Given $x\in L_{\mathcal{A}_q}$, let $P(x)$ be the set of points in $\mathbb{F}_q^n$ which are contained in x, and are not contained in any y such that $y>x$ in $L_{\mathcal{A}_q}$. Then the set x is partitioned by the sets $P(y)$ for $y\geq x$, so we have

$$q^{\dim x} = |x| = \sum_{y\geq x} |P(y)|.$$

By the Möbius inversion formula [33, Proposition 3.7.1] we have

$$|P(x)| = \sum_{y\geq x} \mu(x,y)\, q^{\dim y}.$$

Now, from Theorem 3.3 we know that

$$\begin{aligned} q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t) &= \sum_{x\in L_{\mathcal{A}}} \sum_{p\in P(x)} t^{h(p)} = \sum_{x\in L_{\mathcal{A}}} |P(x)|\, t^{h(x)} \\ &= \sum_{x\leq y \text{ in } L_{\mathcal{A}}} \mu(x,y)\, q^{n-r(y)} t^{h(x)}, \end{aligned}$$

as desired. □

4 Computing coboundary polynomials.

In this section we use Theorem 3.3 to compute the coboundary polynomials of several families of arrangements. As remarked at the beginning of Section 3.1, this is essentially the same as computing their Tutte polynomials.

4.1 Coxeter arrangements.

To illustrate how our finite field method works, we start by presenting some simple examples.

Let Φ be an irreducible crystallographic root system in $\mathbb{R}^n$, and let W be its associated Weyl group. The Coxeter arrangement of type W consists of the hyperplanes $(\alpha, x) = 0$ for each $\alpha \in \Phi^+$, with the standard inner product on $\mathbb{R}^n$. See [18] for an introduction to root systems and Weyl groups, and [25, Chapter 6] or [10, Section 2.3] for more information on Coxeter arrangements.

In this section we compute the coboundary polynomials of the Coxeter arrangements of type A_n, B_n and D_n. (The arrangement of type C_n is the same as the arrangement of type B_n.) The best way to state our results is to compute the exponential generating function for the coboundary polynomials of each family.

The following three theorems have never been stated explicitly in the literature in this form. Theorem 4.1 is equivalent to a result of Tutte [39], who computed the Tutte polynomial of the complete graph. It is also an immediate consequence of a more general theorem of Stanley [36, (15)]. Theorems 4.2 and 4.3 are implicit in the work of Zaslavsky [46].

Theorem 4.1 *Let $\mathcal{A}_n$ be the Coxeter arrangement of type A_{n-1} in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$.*[3] *We have*

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\mathcal{A}_n}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!} \right)^q .$$

Proof. For $n \geq 1$ we have that

$$q\, \overline{\chi}_{\mathcal{A}_n}(q,t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}.$$

for all powers of a large enough prime q, according to Theorem 3.3. For each $p \in \mathbb{F}_q^n$, if we let $A_k = \{i \in [n] \,|\, p_i = k\}$ for $0 \leq k \leq q-1$, then $h(p) = \binom{|A_0|}{2} + \cdots + \binom{|A_{q-1}|}{2}$. Thus

$$q\, \overline{\chi}_{\mathcal{A}_n}(q,t) = \sum_{A_0 \cup \cdots \cup A_{q-1} = [n]} t^{\binom{|A_0|}{2} + \cdots + \binom{|A_{q-1}|}{2}}$$

[3] This arrangement is also known as the *braid arrangement.*

where the sum is over all weak ordered q-partitions of $[n]$. The compositional formula for exponential generating functions [37, Proposition 5.1.3], [7] implies the desired result. $\square$

Theorem 4.2 *Let $\mathcal{B}_n$ be the Coxeter arrangement of type B_n in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ and $x_i + x_j = 0$ for $1 \le i < j \le n$, and the hyperplanes $x_i = 0$ for $1 \le i \le n$. We have*

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{B}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\geq 0} t^{\,n^2} \frac{x^n}{n!}\right).$$

Proof. Let q be a power of a large enough prime, and let $s = \frac{q-1}{2}$. Now for each $p \in \mathbb{F}_q^n$, if we let $B_k = \{i \in [n] \,|\, p_i = k \text{ or } p_i = q-k\}$ for $0 \le k \le s$, we have that $h(p) = |B_0|^2 + \binom{|B_1|}{2} + \cdots + \binom{|B_s|}{2}$. Also, given a weak ordered partition $(B_0, \ldots, B_s)$ of $[n]$, there are $2^{|B_1|+\cdots+|B_s|}$ points of p which correspond to it: for each $i \in B_k$ with $k \neq 0$, we get to choose whether p_i is equal to k or to $q-k$. Therefore

$$q\,\overline{\chi}_{\mathcal{B}_n}(q,t) = \sum_{B_0\cup\,\cdots\,\cup B_s=[n]} t^{|B_0|^2}\left(2^{|B_1|}t^{\binom{|B_1|}{2}}\right)\cdots\left(2^{|B_s|}t^{\binom{|B_s|}{2}}\right),$$

and the compositional formula for exponential generating functions implies Theorem 4.2. $\square$

Theorem 4.3 *Let $\mathcal{D}_n$ be the Coxeter arrangement of type D_n in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ and $x_i + x_j = 0$ for $1 \le i < j \le n$. We have*

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{D}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\geq 0} t^{\,n(n-1)} \frac{x^n}{n!}\right).$$

We omit the details of the proof of Theorem 4.3, which is very similar to the proof of Theorem 4.2.

Setting $t = 0$ in Theorems 4.1, 4.2 and 4.3, it is easy to recover the well-known formulas for the characteristic polynomials of the above arrangements:

$$\begin{aligned}
\chi_{\mathcal{A}_n}(q) &= q(q-1)(q-2)\cdots(q-n+1),\\
\chi_{\mathcal{B}_n}(q) &= (q-1)(q-3)\cdots(q-2n+1),\\
\chi_{\mathcal{D}_n}(q) &= (q-1)(q-3)\cdots(q-2n+3)(q-n+1).
\end{aligned}$$

4.2 Two more examples.

Theorem 4.4 *Let $\mathcal{A}_n^{\#}$ be a generic deformation of the arrangement $\mathcal{A}_n$, consisting of the hyperplanes $x_i - x_j = a_{ij}$ $(1 \le i < j \le n)$, where the a_{ij} are generic real numbers* [4]. *For $n \ge 1$,*

$$q\,\overline{\chi}_{\mathcal{A}_n^{\#}}(q,t) = \sum_F q^{n-e(F)}(t-1)^{e(F)}$$

where the sum is over all forests F on $[n]$, and $e(F)$ denotes the number of edges of F. Also,

$$1 + q\sum_{n\ge 1} \overline{\chi}_{\mathcal{A}_n^{\#}}(q,t)\frac{x^n}{n!} = \left(\sum_{n\ge 0} f(n)\frac{x^n(t-1)^n}{n!}\right)^{\frac{q}{t-1}},$$

where $f(n)$ is the number of forests on $[n]$.

Proof. It is possible to prove Theorem 4.4 using our finite field method, as we did in the previous section. However, it will be easier to proceed directly from (3.2), the definition of the coboundary polynomial.

To each subarrangement $\mathcal{B}$ of $\mathcal{A}_n^{\#}$ we can assign a graph $G_{\mathcal{B}}$ on the vertex set $[n]$, by letting edge ij be in $G_{\mathcal{B}}$ if and only if the hyperplane $x_i - x_j = a_{ij}$ is in $\mathcal{B}$. Since the a_{ij}'s are generic, the subarrangement $\mathcal{B}$ is central if and only if the corresponding graph $G_{\mathcal{B}}$ is a forest. For such a $\mathcal{B}$, it is clear that $|\mathcal{B}| = r(\mathcal{B}) = e(G_{\mathcal{B}})$. Hence,

$$\begin{aligned}
\overline{\chi}_{\mathcal{A}_n^{\#}}(q,t) &= \sum_{\substack{\mathcal{B}\subseteq \mathcal{A}_n^{\#} \\ \text{central}}} q^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|} \\
&= \sum_F q^{(n-1)-e(F)}(t-1)^{e(F)},
\end{aligned}$$

proving the first claim. Now let $c(F) = n - e(F)$ be the number of connected components of F. We have

$$\begin{aligned}
1 + q\sum_{n\ge 1} \overline{\chi}_{\mathcal{A}_n^{\#}}(q,t)\frac{x^n}{n!} &= \sum_{n\ge 0}\sum_{F \text{ on } [n]} \left(\frac{q}{t-1}\right)^{c(F)} \frac{x^n(t-1)^n}{n!} \\
&= \left(\sum_{n\ge 0} f(n)\frac{x^n(t-1)^n}{n!}\right)^{\frac{q}{t-1}}
\end{aligned}$$

by the compositional formula for exponential generating functions. $\square$

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[4] The a_{ij} are "generic" if no n of the hyperplanes have a non-empty intersection, and any non-empty intersection of k hyperplanes has rank k. This can be achieved, for example, by requiring that the a_{ij}'s are linearly independent over the rational numbers. Almost all choices of the a_{ij}'s are generic.

Theorem 4.5 *The threshold arrangement $\mathcal{T}_n$ in $\mathbb{R}^n$ consists of the hyperplanes $x_i + x_j = 0$, for $1 \leq i < j \leq n$. For all $n \geq 0$ we have*

$$\overline{\chi}_{\mathcal{T}_n}(q,t) = \sum_G q^{bc(G)}(t-1)^{e(G)},$$

where the sum is over all graphs G on $[n]$. Here $bc(G)$ is the number of connected components of G which are bipartite, and $e(G)$ is the number of edges of G. Also,

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{T}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}t^{k(n-k)}\frac{x^n}{n!}\right)^{\frac{q-1}{2}}\left(\sum_{n\geq 0} t^{\binom{n}{2}}\frac{x^n}{n!}\right).$$

Proof. Once again, the proof of the first claim is easier using the definition of the coboundary polynomial. Every subarrangement $\mathcal{B}$ of $\mathcal{T}_n$ is central, and we can assign to it a graph $G_{\mathcal{B}}$ as in the proof of Theorem 4.4. In view of (3.2), we only need to check that $r(\mathcal{B}) = n - bc(G_{\mathcal{B}})$ and $|\mathcal{B}| = e(G_{\mathcal{B}})$. The second claim is trivial. To prove the first one, we show that $\dim(\cap\mathcal{B}) = bc(G_{\mathcal{B}})$.

Consider a point p in $\cap\mathcal{B}$. We know that, if ab is an edge in $G_{\mathcal{B}}$, then $p_a = -p_b$. If vertex i is in a connected component C of $G_{\mathcal{B}}$, then the value of p_i determines the value of p_j for all j in C: $p_j = p_i$ if there is a path of even length between i and j, and $p_j = -p_i$ if there is a path of odd length between i and j. If C is bipartite, this determines the values of the p_j's consistently. If C is not bipartite, take a cycle of odd length and a vertex k in it. We get that $p_k = -p_k$, so $p_k = 0$; therefore we must have $p_j = 0$ for all $j \in C$.

Therefore, to specify a point p in $\cap\mathcal{B}$, we split $G_{\mathcal{B}}$ into its connected components. We know that $p_i = 0$ for all i in connected components which are not bipartite. To determine the remaining coordinates of p we have to specify the value of p_j for exactly one j in each bipartite connected component. Therefore $\dim(\cap\mathcal{B}) = bc(G_{\mathcal{B}})$, as desired.

From this point, it is possible to prove the second claim of Theorem 4.5 using the compositional formula for exponential generating functions, in the same way that we proved Theorem 4.4. However, the work involved is considerable, and it is much simpler to use our finite field method, Theorem 3.3, in this case. The proof that we obtain is very similar to the proofs of Theorems 4.1, 4.2 and 4.3, so we omit the details. □

4.3 Deformations of the braid arrangement.

A *deformation of the braid arrangement* is an arrangement in $\mathbb{R}^n$ consisting of the hyperplanes $x_i - x_j = a_{ij}^{(1)}, \ldots, a_{ij}^{(k_{ij})}$ for $1 \leq i < j \leq n$, where the k_{ij} are non-negative integers, and the $a_{ij}^{(r)}$ are real numbers. Such arrangements have been studied extensively by Athanasiadis [6] and Postnikov and Stanley [28]. In this section we study their coboundary polynomials.

Theorem 4.6 *Let $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots)$ be a sequence of arrangements satisfying the following properties:* [5]

[5] Such a sequence is called an *exponential sequence of arrangements.*

1. *$\mathcal{E}_n$ is an arrangement in $\Bbbk^n$, for some fixed field $\Bbbk$.*

2. *Every hyperplane in $\mathcal{E}_n$ is parallel to some hyperplane in the braid arrangement $\mathcal{A}_n$.*

3. *For any subset S of $[n]$, the subarrangement $\mathcal{E}_n^S \subseteq \mathcal{E}_n$, which consists of the hyperplanes in $\mathcal{E}_n$ of the form $x_i - x_j = c$ with $i, j \in S$, is isomorphic to the arrangement $\mathcal{E}_{|S|}$.*

Then

$$1 + q\sum_{n\geq 1} \overline{\chi}_{\mathcal{E}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} \overline{\chi}_{\mathcal{E}_n}(1,t)\frac{x^n}{n!}\right)^q .$$

The special case $t = 0$ of this result is due to Stanley [34, Theorem 1.2]; we omit the proof, which is an easy extension of his.

The most natural examples of exponential sequences of arrangements are the following. Fix a set A of k distinct integers $a_1 < \ldots < a_k$. Let $\mathcal{E}_n$ be the arrangement in $\mathbb{R}^n$ consisting of the hyperplanes

$$x_i - x_j = a_1, \ldots, a_k \qquad 1 \leq i < j \leq n. \tag{4.1}$$

Then $(\mathcal{E}_0, \mathcal{E}_1, \ldots)$ is an exponential sequence of arrangements and Theorem 4.6 applies to this case. In fact, we can say much more about this type of arrangement.

After proving the results in this section, we found out that Postnikov and Stanley [28] had used similar techniques in computing the characteristic polynomials of these types of arrangements. Therefore, for consistency, we will use the terminology that they introduced.

Definition 4.7 *A* graded graph *is a triple $G = (V_G, E_G, h_G)$, where V_G is a linearly ordered set of vertices (usually $V_G = [n]$), E_G is a set of (non-oriented) edges, and h_G is a function $h_G : V \to \mathbb{N}$, called a* grading.

We will drop the subscripts when the underlying graded graph is clear. We will refer to $h(v)$ as the *height* of vertex v. The *height* of G, denoted $h(G)$, is the largest height of a vertex of G.

Definition 4.8 *Let G be a graded graph and r be a non-negative integer. The* r-th level *of G is the set of vertices v such that $h(v) = r$. G is* planted *if each one of its connected components has a vertex on the* 0*-th level.*

Definition 4.9 *If $u < v$ are connected by edge e in a graded graph G, the* slope *of e is $s(e) = h(u) - h(v)$. G is an* A-graph *if the slopes of all edges of G are in $A = \{a_1, \ldots, a_k\}$.*

Recall that, for a graph G, we let $e(G)$ be the number of edges and $c(G)$ be the number of connected components of G. We also let $v(G)$ be the number of vertices of G.

Proposition 4.10 *Let $\mathcal{E}_n$ be the arrangement (4.1). Then, for $n \geq 1$,*

$$q\,\overline{\chi}_{\mathcal{E}_n}(q,t) = \sum_G q^{c(G)}(t-1)^{e(G)},$$

where the sum is over all planted graded A-graphs on $[n]$.

Proof. We associate to each planted graded A-graph $G = (V, E, h)$ on $[n]$ a central subarrangement $\mathcal{A}_G$ of $\mathcal{E}_n$. It consists of the hyperplanes $x_i - x_j = h(i) - h(j)$, for each $i < j$ such that ij is an edge in G. This is a subarrangement of $\mathcal{E}_n$ because $h(i) - h(j)$, the slope of edge ij, is in A. It is central because the point $(h(1), \ldots, h(n)) \in \mathbb{R}^n$ belongs to all these hyperplanes.

This is in fact a bijection between planted graded A-graphs on $[n]$ and central subarrangements of $\mathcal{E}_n$. To see this, take a central subarrangement $\mathcal{A}$. We will recover the planted graded A-graph G that it came from. For each pair (i, j) with $1 \leq i < j \leq n$, $\mathcal{A}$ can have at most one hyperplane of the form $x_i - x_j = a_t$. If this hyperplane is in $\mathcal{A}$, we must put edge ij in G, and demand that the heights $h(i)$ and $h(j)$ satisfy $h(i) - h(j) = a_t$. When we do this for all the hyperplanes in $\mathcal{A}$, the height requirements that we introduce are consistent, because $\mathcal{A}$ is central. However, these requirements do not fully determine the heights of the vertices; they only determine the relative heights within each connected component of G. Since we want G to be planted, we demand that the vertices with the lowest height in each connected component of G should have height 0. This does determine G completely, and clearly $\mathcal{A} = \mathcal{A}_G$.

Example. Consider an arrangement $\mathcal{E}_8$ in $\mathbb{R}^8$, with a subarrangement consisting of the hyperplanes $x_1 - x_2 = 4, x_1 - x_3 = -1, x_1 - x_6 = 0, x_1 - x_8 = 1, x_2 - x_3 = -5$ and $x_4 - x_7 = 2$. Figure 1 shows the planted graded A-graph corresponding to this subarrangement.

With this bijection in hand, and keeping (3.2) in mind, it remains to show that $r(\mathcal{A}_G) = n - c(G)$ and $|\mathcal{A}_G| = e(G)$. The second of these claims is trivial. We omit the proof of the first one which is very similar to, and simpler than, that of $r(\mathcal{B}) = n - bc(G_{\mathcal{B}})$ in our proof of Theorem 4.5. □

Theorem 4.11 *Let $\mathcal{E}_n$ be the arrangement (4.1), and let*

$$A_r(t,x) = \sum_{n\geq 0} \left(\sum_{f:[n]\to[r]} t^{a(f)} \right) \frac{x^n}{n!}, \tag{4.2}$$

where $a(f)$ denotes the number of pairs (i, j) with $1 \leq i < j \leq n$ such that $f(i) - f(j) \in A$. Then

$$1 + q \sum_{n\geq 1} \overline{\chi}_{\mathcal{E}_n}(q,t) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(t,x)}{A_{r-1}(t,x)} \right)^q. \tag{4.3}$$

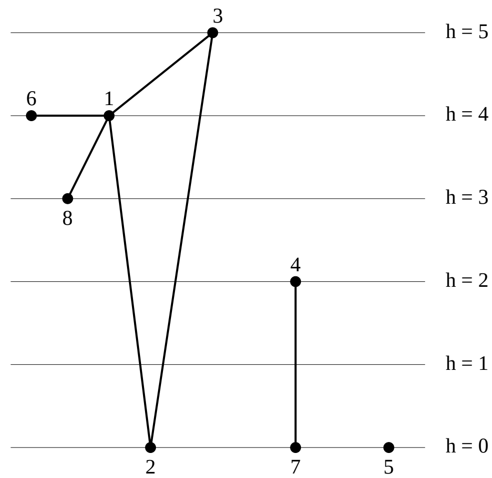

Figure 1: The planted graded A-graph corresponding to a subarrangement of $\mathcal{E}_8$.

Remark. The limit in (4.3) is a limit in the sense of convergence of formal power series. Let $F_1(t,x), F_2(t,x), \dots$ be a sequence of formal power series. We say that $\lim_{n\to\infty} F_n(t,x) = F(t,x)$ if, for all a and b, the coefficient of $t^a x^b$ in $F_n(t,x)$ is equal to the coefficient of $t^a x^b$ in $F(t,x)$ for all n larger than some constant $N(a,b)$. For more information on this notion of convergence, see [33, Section 1.1] or [23].

Proof of Theorem 4.11. First we prove that

$$A_r(t,x) = \sum_G (t-1)^{e(G)} \frac{x^{v(G)}}{v(G)!} \tag{4.4}$$

where the sum is over all graded A-graphs G of height less than r. The coefficient of $\frac{x^n}{n!}$ in the right-hand side of (4.4) is $\sum_G (t-1)^{e(G)}$, summing over all graded A-graphs G on $[n]$ with height less than r. We have

$$\begin{aligned}
\sum_G (t-1)^{e(G)} &= \sum_{h:[n]\to[0,r-1]} \ \sum_{\substack{G \text{ such that} \\ h_G = h}} (t-1)^{e(G)} \\
&= \sum_{h:[n]\to[0,r-1]} (1+(t-1))^{a(h)} \\
&= \sum_{f:[n]\to[r]} t^{a(f)}
\end{aligned}$$

The only tricky step here is the second: if we want all graded A-graphs G on $[n]$ with a specified grading h, we need to consider the possible choices of edges of the graph. Any edge ij can belong to the graph, as long as $h(i) - h(j) \in A$, so there are $a(h)$ possible edges.

Equation (4.4) suggests the following definitions. Let

$$B_r(t,x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded A-graphs G of height less than r, and let

$$B(t,x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded A-graphs G.

The equation

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\mathcal{E}_n}(q,t) \frac{x^n}{n!} = B(t-1,x)^q, \tag{4.5}$$

follows from Proposition 4.10, using either Theorem 4.6 or the compositional formula for exponential generating functions.

Now we claim that $B(t,x) = \lim_{r\to\infty} B_r(t,x)$. Notice that, in a planted graded A-graph G with e edges and v vertices, each vertex has a path of length at most v which connects it to a vertex on the 0-th level. Recalling that $a_1 < \ldots < a_k$ we see that $h(G) \leq v \cdot \max(-a_1, a_k)$, so the coefficients of $t^e \frac{x^v}{v!}$ in $B_r(t,x)$ and $B(t,x)$ are equal for $r > v \cdot \max(-a_1, a_k)$.

With a little bit of care, it then follows easily that

$$B(t-1,x) = \lim_{r\to\infty} B_r(t-1,x). \tag{4.6}$$

Here it is necessary to check that $B(t-1,x)$ is, indeed, a formal power series. This follows from the observation that the coefficient of $\frac{x^n}{n!}$ in $B(t,x)$ is a polynomial in t of degree at most $\binom{n}{2}$. We know that for some formal power series $f(t)$ (like e^t, for example), $f(t-1)$ is not a well-defined formal power series. In our case, however, this is not a problem and (4.6) is valid. Once again, see [33, Section 1.1] for more information on these technical details.

Next, we show that

$$B_r(t-1,x) = A_r(t,x)/A_{r-1}(t,x) \tag{4.7}$$

or, equivalently, that $A_r(t,x) = B_r(t-1,x)A_{r-1}(t,x)$. The multiplication formula for exponential generating functions ([37, Proposition 5.1.1]) and (4.4) give us a combinatorial interpretation of this identity. We need to show that the ways of putting the structure of a graded A-graph G with $h(G) < r$ on $[n]$ can be put in correspondence with the ways of doing the following: first splitting $[n]$ into two disjoint sets S_1 and S_2, then putting the structure of a *planted* graded A-graph G_1 with $h(G_1) < r$ on S_1, and then putting the structure of a graded A-graph G_2 with $h(G_2) < r-1$ on S_2. We also need that, in that correspondence, $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$.

We do this as follows. Let G be a graded A-graph G with $h(G) < r$. Let G_1 be the union of the connected components of G which contain a vertex on the

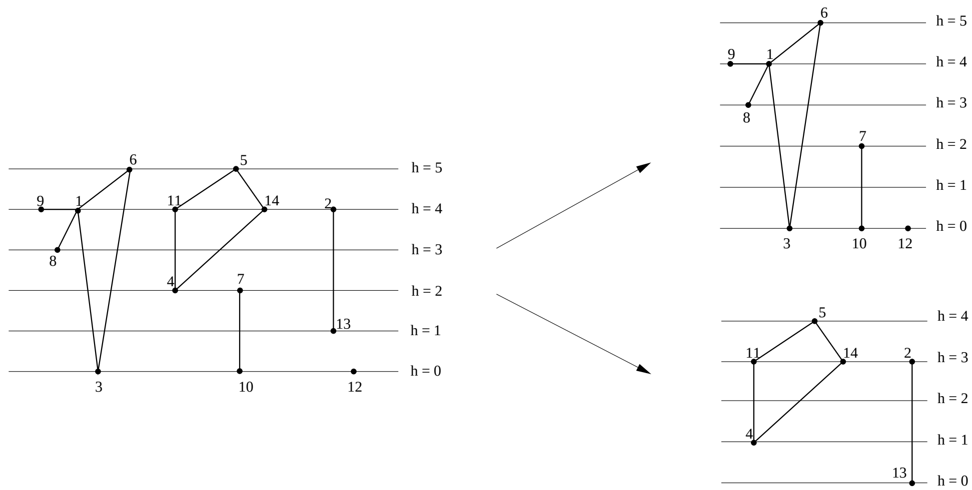

Figure 2: The decomposition of a graded A-graph.

0-th level. Put a grading on G_1 by defining $h_{G_1}(v) = h_G(v)$ for $v \in G_1$. Let $G_2 = G - G_1$. It is clear that $h_G(v) \geq 1$ for all $v \in G_2$; therefore we can put a grading on G_2 by defining $h_{G_2}(v) = h_G(v) - 1$ for $v \in G_2$. G_1 is a planted graded A-graph with $h(G_1) < r$, and G_2 is a graded A-graph with $h(G_2) < r-1$. Figure 2 illustrates this decomposition with an example.

Our map from G to a pair (G_1, G_2) is a one-to-one correspondence. Any pair (G_1, G_2), with G_1 planted of height less than r and G_2 of height less than $r-1$, arises from a decomposition of some G of height less than r in this way. It is clear how to recover G from G_1 and G_2. Also, it is clear from the construction of the correspondence that $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$. This completes the proof of (4.7).

Now we just have to put together (4.5), (4.6) and (4.7) to complete the proof of Theorem 4.11. $\square$

The *Catalan arrangement* C_n in $\mathbb{R}^n$ consists of the hyperplanes

$$x_i - x_j = -1, 0, 1 \qquad 1 \leq i < j \leq n.$$

When the arrangement in Theorem 4.11 is a subarrangement of the Catalan arrangement, we can say more about the power series A_r of (4.2). Let

$$A(t,x,y) = \sum_r A_r(t,x) y^r = \sum_{n \geq 0} \sum_{r \geq 0} \left(\sum_{f:[n]\to[r]} t^{a(f)} \right) \frac{x^n}{n!} y^r$$

and let

$$S(t,x,y) = \sum_{n \geq 0} \sum_{r \geq 0} \left(\sum_{f:[n]\to[r]} t^{a(f)} \right) \frac{x^n}{n!} y^r \tag{4.8}$$

where the inner sum is over all *surjective* functions $f : [n] \to [r]$. The following proposition reduces the computation of $A(t, x, y)$ to the computation of $S(t, x, y)$, which is often easier in practice.

Proposition 4.12 *If $A \subseteq \{-1, 0, 1\}$ in the notation of Theorem 4.11, we have*

$$A(t, x, y) = \frac{S(t, x, y)}{1 - yS(t, x, y)}$$

Proof. Once again, we think of this as an identity about exponential generating functions in the variable x. Fix n, r, and $f : [n] \to [r]$. Let the image of f be $\{1, \ldots, m_1 - 1\} \cup \{m_1 + 1, \ldots, m_1 + m_2 - 1\} \cup \cdots \cup \{m_1 + \cdots + m_{k-1} + 1, \ldots, m_1 + \cdots + m_k - 1\} = M_1 \cup \cdots \cup M_k$, so that $[r] - \operatorname{Im} f = \{m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_{k-1}\}$. Here $m_1, \ldots, m_k$ are arbitrary positive integers such that $m_1 + \cdots + m_k - 1 = r$. For $1 \leq i \leq k$, let f_i be the restriction of f to $f^{-1}(M_i)$; it maps $f^{-1}(M_i)$ surjectively to M_i. Then we can "decompose" f in a unique way into the k *surjective* functions $f_1, \ldots, f_k$. The weight $w(f)$ corresponding to f in $A(t, x, y)$ is $t^{a(f)} y^r$, while the weight $w(f_i)$ corresponding to f_i in $S(t, x, y)$ is $t^{a(f_i)} y^{m_i - 1}$.

Now observe that $a(f) = a(f_1) + \cdots + a(f_k)$: whenever we have a pair of numbers $1 \leq i < j \leq n$ counted by $a(f)$, since $f(i) - f(j) \in \{-1, 0, 1\}$, we know that $f(i)$ and $f(j)$ must be in the same M_h. Therefore i and j are in the same $f^{-1}(M_h)$, and this pair is also counted in $a(f_h)$. We also have that $r = (m_1 - 1) + \cdots + (m_k - 1) + (k - 1)$. Therefore $w(f) = w(f_1) \cdots w(f_k) y^{k-1}$. It follows from the compositional formula for exponential generating functions that

$$\begin{aligned} A(t, x, y) &= \sum_{k \geq 1} S(t, x, y)^k y^{k-1} \\ &= \frac{S(t, x, y)}{1 - yS(t, x, y)} \end{aligned}$$

as desired. □

Considering the different subsets of $\{-1, 0, 1\}$, we get six non-isomorphic subarrangements of the Catalan arrangement. They come from the subsets $\emptyset$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{-1, 1\}$ and $\{-1, 0, 1\}$. The corresponding subarrangements are the empty arrangement, the braid arrangement, the Linial arrangement, the Shi arrangement, the semiorder arrangement, and the Catalan arrangement, respectively. The empty arrangement is trivial, and the braid arrangement was already treated in detail in Section 4.1. We now have a technique that lets us talk about the remaining four arrangements under the same framework. We will do this in the remainder of this chapter.

4.3.1 The Linial arrangement

The Linial arrangement $\L_n$ consists of the hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$. This arrangement was first considered by Linial and Ravid. It was later

studied by Athanasiadis [4] and Postnikov and Stanley [28], who independently computed the characteristic polynomial of Ł_n:

$$\chi_{\text{Ł}_n}(q) = \frac{q}{2^n} \sum_{k=0}^{n} \binom{n}{k} (q-k)^{n-1}.$$

They also put the regions of Ł_n in bijection with several different sets of combinatorial objects. Perhaps the simplest such set is the set of *alternating trees* on $[n+1]$: the trees such that every vertex is either larger or smaller than all its neighbors.

Now we present the consequences of Proposition 4.10, Theorem 4.11 and Proposition 4.12 for the Linial arrangement. Recall that a poset P on $[n]$ is *naturally labeled* if $i < j$ in P implies $i < j$ in $\mathbb{Z}^+$.

Proposition 4.13 *For all $n \geq 1$ we have*

$$q\,\overline{\chi}_{\text{Ł}_n}(q,t) = \sum_{P} q^{c(P)}(t-1)^{e(P)}$$

where the sum is over all naturally labeled, graded posets P on $[n]$. Here $c(P)$ and $e(P)$ denote the number of components and edges of the Hasse diagram of P, respectively.

Proof. There is an obvious bijection between Hasse diagrams of naturally labeled graded posets on $[n]$ and planted graded $\{1\}$-graphs on $[n]$. The result then follows immediately from Proposition 4.10. □

Theorem 4.14 *Let*

$$A_r(t,x) = \sum_{n \geq 0} \left(\sum_{f:[n]\to[r]} t^{id(f)} \right) \frac{x^n}{n!}.$$

where $id(f)$ denotes the number of inverse descents *of the word $f(1)\dots f(n)$: the number of pairs (i,j) with $1 \leq i < j \leq n$ such that $f(i) - f(j) = 1$. Then*

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\text{Ł}_n}(q,t) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(t,x)}{A_{r-1}(t,x)} \right)^q.$$

Proof. This is immediate from Theorem 4.11. □

Recall that the *descents* of a permutation $\sigma = \sigma_1 \dots \sigma_r \in S_r$ are the indices i such that $\sigma_i > \sigma_{i+1}$. For more information about descents, see for example [33, Section 1.3].

We call $id(f)$ the number of inverse descents, because they generalize descents in the following way. If $\pi : [r] \to [r]$ is a permutation, then $id(\pi)$ is the number of descents of the permutation π^{-1}. If, similarly, we consider the list

of sets $f^{-1}(1), \ldots, f^{-1}(r)$, then $id(f)$ counts the number of occurrences of an $x \in f^{-1}(i)$ and a $y \in f^{-1}(i+1)$ such that $x > y$.

It would be nice to compute the polynomials $A_r(t,x)$ above explicitly. We have not been able to do this. However, the special case $t = 0$ is of interest; recall that the characteristic polynomial of $\mathrm{Ł}_n$ is $\chi_{\mathrm{Ł}_n}(q) = q\overline{\chi}_{\mathrm{Ł}_n}(q,0)$. In that case, we obtain the following result.

Theorem 4.15 *Let*

$$\frac{1+ye^{x(1+y)}}{1-y^2e^{x(1+y)}} = \sum_{r\geq 0} A_r(x)y^r. \tag{4.9}$$

Then we have

$$\sum_{n\geq 0} \chi_{Ł_n}(q)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)}\right)^q.$$

In particular, if f_n is the number of alternating trees on $[n+1]$, we have

$$\sum_{n\geq 0}(-1)^n f_n \frac{x^n}{n!} = \lim_{r\to\infty}\frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. In view of Theorem 4.14 and Proposition 4.12, we compute $S(0,x,y)$. From (4.8), the coefficient of $\frac{x^n}{n!}y^r$ in $S(0,x,y)$ is equal to the number of surjective functions $f : [n] \to [r]$ with no inverse descents. These are just the non-decreasing surjective functions $f : [n] \to [r]$. For $n \geq 1$ and $r \geq 1$ there are $\binom{n-1}{r-1}$ such functions, and for $n = r = 0$ there is one such function. In the other cases there are none. Therefore

$$\begin{aligned} S(0,x,y) &= 1 + \sum_{n\geq 1}\sum_{r\geq 1}\binom{n-1}{r-1}\frac{x^n}{n!}y^r \\ &= 1 + \sum_{n\geq 1}\frac{x^n}{n!}y(1+y)^{n-1} \\ &= \frac{1+ye^{x(1+y)}}{1+y}. \end{aligned}$$

Proposition 4.12 then implies that

$$A(0,x,y) = \frac{1+ye^{x(1+y)}}{1-y^2e^{x(1+y)}},$$

in agreement with (4.9), and the theorem follows. □

4.3.2 The Shi arrangement

The Shi arrangement $\mathcal{S}_n$ consists of the hyperplanes $x_i - x_j = 0, 1$ for $1 \leq i < j \leq n$. Shi ([31, Chapter 7], [32]) first considered this arrangement, and showed

that it has $(n+1)^{n-1}$ regions. Headley ([16, Chapter VI], [17]) later computed the characteristic polynomial of $\mathcal{S}_n$:

$$\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}.$$

Stanley [34],[35] gave a nice bijection between regions of the Shi arrangement and parking functions of length n. Parking functions were first introduced by Konheim and Weiss [20]; for more information about them, see [37, Exercise 5.49].

For the Shi arrangement, we can say the following.

Theorem 4.16 *Let*

$$A_r(x) = \sum_{n=0}^{r} (r-n)^n \frac{x^n}{n!}.$$

Then we have

$$\sum_{n \geq 0} \chi_{\mathcal{S}_n}(q) \frac{x^n}{n!} = \left(\lim_{r \to \infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular, we have

$$\sum_{n \geq 0} (-1)^n (n+1)^{n-1} \frac{x^n}{n!} = \lim_{r \to \infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. We proceed in the same way that we did in Theorem 4.15. In this case, we need to compute the number of surjective functions $f : [n] \to [r]$ such that $f(i) - f(j)$ is never equal to 0 or 1 for $i < j$. These are just the surjective, strictly increasing functions. There is only one of them when $n = r$, and there are none when $n \neq r$. Hence

$$S(0, x, y) = \sum_{n \geq 0} \frac{x^n}{n!} y^n = e^{xy}.$$

The rest follows easily by computing $A(0, x, y)$ and $A_r(x)$ explicitly. $\square$

4.3.3 The semiorder arrangement

The semiorder arrangement $\mathcal{I}_n$ consists of the hyperplanes $x_i - x_j = -1, 1$ for $1 \leq i < j \leq n$. A *semiorder on* $[n]$ is a poset P on $[n]$ for which there exist n unit intervals $I_1, \ldots, I_n$ of $\mathbb{R}$, such that $i < j$ in P if and only if I_i is disjoint from I_j and to the left of it. It is known [30] that a poset is a semiorder if and only if it does not contain a subposet isomorphic to $\mathbf{3} + \mathbf{1}$ or $\mathbf{2} + \mathbf{2}$. We are interested in semiorders because the number of regions of $\mathcal{I}_n$ is equal to the number of semiorders on $[n]$, as shown in [28] and [34].

Theorem 4.17 *Let*

$$\frac{1 - y + ye^x}{1 - y + y^2 - y^2 e^x} = \sum_{r \geq 0} A_r(x) y^r.$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{I}_n}(q)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)}\right)^q.$$

In particular, if i_n is the number of semiorders on $[n]$, we have

$$\sum_{n\geq 0} (-1)^n i_n \frac{x^n}{n!} = \lim_{r\to\infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. In this case, $S(0,x,y)$ counts surjective functions $f:[n]\to[r]$ such that $f(i)-f(j)$ is never equal to 1 for $i\neq j$. Such a function has to be constant; so it can only exist (and is unique) if $n\geq 1$ and $r=1$ or if $n=r=0$. Thus

$$S(0,x,y) = 1+(e^x-1)y$$

and the rest follows easily. □

4.3.4 The Catalan arrangement

The Catalan arrangement C_n consists of the hyperplanes $x_i - x_j = -1, 0, 1$ for $1\leq i<j\leq n$. Stanley [34] observed that the number of regions of this arrangement is $n!C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. For (much) more information on the Catalan numbers, see [37, Chapter 6], especially Exercise 6.19.

Theorem 4.18 *Let*

$$A_r(x) = \sum_{n=0}^{\lfloor \frac{r+1}{2}\rfloor} \binom{r-n+1}{n} x^n.$$

Then we have

$$\sum_{n\geq 0} \chi_{C_n}(q)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)}\right)^q.$$

In particular,

$$\frac{\sqrt{1+4x}-1}{2x} = \sum_{n\geq 0}(-1)^n C_n x^n = \lim_{r\to\infty}\frac{A_{r-1}(x)}{A_r(x)}. \tag{4.10}$$

Proof. There are no surjective functions $f:[n]\to[r]$ such that $f(i)-f(j)$ is never equal to $-1, 0$ or 1 for $i\neq j$, unless $n=r=0$ or $n=r=1$. Thus $S(x,y,0)=1+xy$. The rest of the proof is easy. □

The polynomial $A_r(x)$ is a simple transformation of the *Fibonacci polynomial*. The number of words of length r, consisting of 0's and 1's, which do not contain two consecutive 1's, is equal to F_{r+2}, the $(r+2)$-th Fibonacci number. It is easy to see that the polynomial $A_r(x)$ counts those words according to the number of 1's they contain. In particular, $A_r(1)=F_{r+2}$.

We close with an amusing observation. Irresponsibly plugging $x = 1$ into (4.10) [6], we obtain an unconventional "proof" of the rate of growth of Fibonacci numbers:

$$\frac{\sqrt{5}-1}{2} = \lim_{r\to\infty} \frac{F_{r-1}}{F_r}.$$

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